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ON CONSTRUCTING THE CONTOUR OF MINIMUM WAVE DRAG IN AN INHOMOGENEOUS SUPERSONIC FLOW

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A variational problem is considered of constructing the generatrix of a plane or axisymmetric body guaranteeing the minimum wave drag in an inhomogeneous (nonisentropic and nonisoenergetic) supersonic flow of an ideal gas (inviscid and non-heat-conducting) in the case when the domain of determinacy of the unknown contour contains a zone of sharp variation in the values of parameters which are retained (in the absence of jumps) along the streamline, the parameters being the entropy and the stagnation enthalpy. In the limit the zone degenerates to a tangential discontinuity. The investigation is limited to the configurations (e. g. nozzles or the stern parts of the bodies) for which no shock waves (this includes the bow shock) exist in the region under investigation. It is established that the solution [1, 2] obtained earlier for inhomogeneous flows and yielding a smooth optimal countour (without internal corner points) cannot be realized in such cases and must be replaced by a solution in which the generatrix of the optimal body contains at least one internal corner point. Since the method of passing to the control contour utilized in [1, 2] cannot be applied to the study of such configurations, the necessary extremal conditions determining the form of the optimal generatrix must be obtained using the general method of Lagrange mulipliers in the form developed in [3 – 5]. The conditions of optimal-

ity obtained are used to derive a numerical algorithm, and examples of the optimal generatrices of plane bodies are given for the case of a flow with a tangential discontinuity.

1. We consider the problem of constructing a generatrix ag of a plane ($\nu = 0$) or axisymmetric ($\nu = 1$) body, guaranteeing a minimum wave drag in a supersonic flow of an ideal gas. Let the gas flow from left to right and let the axes of a rectangular coordinate system xy which, in the axisymmetric case, lies in the meridional plane, be placed so that the initial point a of the segment of the generatrix sought lies on the y-axis, as shown in Fig. 1 (for $\nu = 1$ the x-axis is the axis of symmetry). The geometrical characteristics which are assumed given are, in addition to the coordinates of the point a, the maximum allowable length X of the body under consideration and y_g which is the value of the ordinate of the end point of the contour (here and in the following the subscripts a, b, g, ... indicate parameters at the corresponding points). Since the optimal contour is limited in length, it may contain the butt end bg which is the part of the extremal edge at which $x \equiv X$, and the allowed variations $\delta x \leqslant 0$. The butt end is not exposed to the gas flow and the pressure p^+ acting on it will be assumed as a known constant independent of the form of the contour sought.

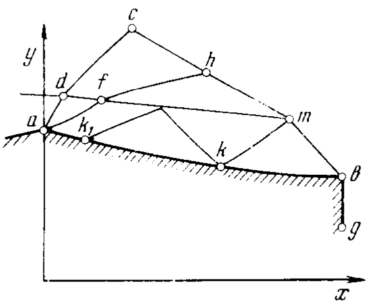

Fig. 1

In the general case the direction of the tangent to the contour to the right of the point a does not necessarily agree with the direction of the velocity vector of the free supersonic stream on the left of this point. Therefore the flow past the contour ag is accompanied by formation of either a fan of expansion waves originating at the point a, or an attached or detached shock wave. We shall restrict ourselves to the first possibility which corresponds to a flow e. g. past the afterbody section, past the upper part of the profile at a sufficiently large angle of attack, or past a nozzle contour (the last case will be considered separately, since here it is expedient to conduct the investigation in the upper semiplane where the orientation of the contour ag is different from that shown in Fig. 1). In this case the flow parameters at the characteristic ac which forms the left boundary of the initial expansion fan, can be regarded as known functions of ψ, where ψ is the stream function defined by the condition $\psi_a = 0$ and the equation

$$d\psi = cy^{\nu}\rho\,(udy - vdx) \tag{1.1}$$

Here ρ is the density and c is a normalizing constant chosen arbitrarily, while u and v are the projections of the velocity vector on the x and y axes. Equation (1.1) connects the x, y and ψ increments along an arbitrary curve in the xy plane, and in particular along the characteristic ac.

When the shock waves or any other dissipative processes are absent from the region considered (which is assumed), the specific entropy s of the gas and its stagnation enthalpy $H = h + w^2 / 2$, where h is the specific enthalpy and $w^2 = u^2 + v^2$ is the square of the velocity modulus, are conserved along each streamline. Therefore in the general case of an inhomogeneous (nonisentropic and nonisoenergetic) flow it is expedient to conduct the investigation using the variables ψy in which case the equations describing the flow past the segment ab of the contour sought, have the form

$$L_1 \equiv \frac{\partial u}{\partial y} - \frac{\partial y^\nu p}{\partial \psi} = 0, \qquad L_2 \equiv \frac{\partial (y^\nu \rho v)^{-1}}{\partial y} + \frac{\partial (u / v)}{\partial \psi} = 0 \tag{1.2}$$

$$2h(p, \rho, \psi) + w^2 = 2H(\psi), \qquad s(p, \rho, \psi) = S(\psi)$$

Here p is the pressure, and the functions entering the last two relations are assumed known. These functions may have first order discontinuities in ψ, which correspond, in the flow plane, to the tangential discontinuities coinciding with the streamlines. In general, on passage through such discontinuities all parameters undergo a jump, with the exception of the pressure and the angle $\vartheta \equiv \operatorname{arctg}(v / u)$ formed by the velocity vector and the x-axis. In the following we shall limit ourselves to the case when the domain of definition of the segment ab bounded from the left and right by the characteristics ah and hb is entered by not more than one line of tangential discontinuity. It is this situation that is depicted in Fig. 1, where the line of tangential discontinuity is the steamline dm. The jumps in the values of the Mach number $M = w / a$ (where $a = a(p, \rho, \psi)$ is the speed of sound) and in the value of the Mach angle $\alpha = \arcsin(1 / M)$ observed during the passage along dm, have the corresponding corners on the characteristics (ah is the last boundary characteristic of the initial fan), as shown in Fig. 1.

Let us formulate the variational problem. Under the above conditions we require to construct the generatrix ag, i.e. to determine the relationship $x = \xi(y)$, where $0 \leqslant \xi(y) \leqslant X$, realizing the minimum wave drag χ. The latter is given, with the accuracy to within the unessential term and multiplier, by

$$\chi = \int_a^b p y^\nu \, dy + \frac{y_g^{1+\nu} - y_b^{1+\nu}}{1+\nu} p^+ \tag{1.3}$$

Here integration is performed along the body contour along which the following condition of no-flow holds

$$L_3 \equiv \frac{\partial x}{\partial y} - \frac{u}{v} = 0 \tag{1.4}$$

It is expedient to assume that the parameters in (1.1) – (1.4) are dimensionless. If l_*, w_* and ρ_* are respectively the characteristic length, velocity and density, then we refer the quantities possessing the dimensions of length to l_*, of velocity to w_*, of density to ρ_* and of pressure to $\rho_* w_*^2$, the stream function to $\rho_* w_* l_*^{1+\nu}$ and the wave drag to $\rho_* w_*^2 l_*^{1+\nu}$.

2. We assume that the optimal configuration sought has no internal corner points, i.e. is of the type depicted in Fig. 1. Then the problem can be solved, as in the case of absence of a contact discontinuity [1, 2], using the method of control contour. Taking the stream function ψ as the independent variable on ac, ah and cb and assuming that the value $\psi_m = \psi_l$ at the point m is fixed while the variations $\delta\vartheta$ and δw, by virtue of continuity of ϑ and p, are connected by the relations

$$\delta\vartheta_{m+} = \delta\vartheta_{m-}, \qquad (\rho w)_{m+} \delta w_{m+} = (\rho w)_{m-} \delta w_{m-}$$

where minus (plus) denotes the upper (lower) limiting values at the tangential discontinuity, we arrive at the following results. For the optimal contour ag, the following equality must hold on the characteristic hb (λ is a constant which is the same on both

hm and mb):

$$y^\nu \rho w^2 \operatorname{tg} \alpha \sin^2 \vartheta = \lambda \tag{2.1}$$

This equation (with the point h suitably selected) and the equations

$$\frac{dx}{d\psi} + \frac{u\sqrt{w^2-a^2}+av}{y^\nu \rho a w^2} = 0, \qquad \frac{dy}{d\psi} + \frac{v\sqrt{w^2-a^2}-au}{y^\nu \rho a w^2} = 0$$

$$u^2 \frac{d(v/u)}{d\psi} - \frac{\sqrt{w^2-a^2}}{\rho a}\frac{dp}{d\psi} + \frac{\nu v}{y^{1+\nu}\rho} = 0 \tag{2.2}$$

which hold along any characteristic belonging to the second family, are together sufficient for constructing the characteristic hb. The ordinate of b either coincides with y_g or (for $y_b > y_g$) is found from the Busemann condition

$$(\rho w^2 \operatorname{tg} \alpha \sin 2\vartheta)_b = 2(p^+ - p_b) \tag{2.3}$$

At the first sight it appears that the relations (2.1) – (2.3) together with the arbitrariness in the choice of h, i.e. in the choice of intensity of the initial fan and position of h on its closing characteristic are sufficient to construct the optimal characteristic hb and then (having solved the Coursat problem) the contour ag satisfying all the conditions given. However this is not so. Indeed, since Eq. (2.1) containing the same single constant λ must hold on hb on both sides of the point m, it follows, together with the fact that y and ϑ both are continuous at this point taken into account, that at m the following equation must hold

$$\Delta \equiv 1 - \omega_m = 0$$

$$(\omega = (\rho w^2 \operatorname{tg} \alpha)_+ / (\rho w^2 \operatorname{tg} \alpha)_-)$$

Since ω contains, apart from the thermodynamic parameters, only the velocity modulus, using the last relations of (1.2) we can show that ω is a function of H_+, H_-, S_+, S_- and pressure only (the definition of ω clearly implies that in the absence of a tangential discontinuity when $H_+ = H_-$ and $S_+ = S_-$, $\omega = 1$ for any p). Recalling that the limiting values of H and S on both sides of the tangential discontinuity coincide with the equivalent quantities on ac, i.e. that they are given, we see that in the general case ($H_+ \neq H_-$ and $S_+ \neq S_-$) the condition $\Delta = 0$, even if it holds, can only do so when p_m is a root of the equation

$$\omega(H_+, H_-, S_+, S_-, p_m) = 1 \tag{2.4}$$

Thus, even when the above equation has positive real roots satisfying the flow scheme under consideration, the configuration depicted in Fig. 1 cannot be constructed in the general case.

The physical meaning of the condition (2.4) is easily explained. It can be shown that the coefficient of reflection from the contact discontinuity of the pressure perturbations arriving at m along the characteristic km of the first family and reflected along the characteristic of the opposite family, is

$$K = (1 - \omega) / (1 + \omega)$$

Consequently Eq. (2.4) is equivalent to the condition that the coefficient of reflection vanishes and this, as a rule, does not take place.

The fact that for $K \neq 0$ the configuration containing no internal corner points cannot be optimal, may be proved using an example proposed in [6] in the investigation of a flow past bodies of approximately wedge shape. Repeating this procedure (see also [7]) we vary the contour without a corner at the point k only over the interval $(y_k - \Delta y) < y < (y_k + \Delta y)$, where Δy is a small positive quantity. Within this interval we replace the initial contour (as shown in Fig. 2) by two rectilinear segments intersecting the initial contour at the interval boundaries, and each other at a point on the unvaried characteristic km the abscissa of which is $x_k + \Delta y \Delta\zeta$, where $\zeta = v / u$ and $\Delta\zeta$ is of the same order of magnitude as Δy. Linearizing the equations of motion relative to the original (nonuniform) stream, it can be shown that within the accuracy of order higher than that of $\Delta y \Delta\zeta$, the perturbations in p caused by the process of varying the contour performed above, vanish everywhere outside the strips of height Δy adjacent to the characteristics km and mb. When the reflection coefficient is positive $(K_m > 0)$ the pressure increases (decreases) in the regions denoted in Fig. 2 by the plus (minus) signs. It can further be shown that the increment in the value of χ caused by the variation of p on the altered segment of the contour has also a higher order of smallness than $\Delta y \Delta\zeta$. Thus if $K_m > 0$ and $\vartheta_b < 0$, we are left only with the uncompensated negative increment in χ of the order of $\Delta y \Delta\zeta$. When $K_m < 0$, we obtain the same result with $\Delta\zeta < 0$, which corresponds to an indentation near the point k. From this we see that for $K_m \neq 0$, the contour without a corner at the point k cannot be optimal. Moreover, the above analysis enables us to assert that when $K_m > 0$, the optimal configuration has a corner at the point indicated and an expansion fan is formed in the flow past this corner, while for $K_m < 0$ we have a shock wave. Similar analysis can be performed in the case when the segment ak contains another one (k_1 in Fig. 1) or several points such, that the perturbations generated by deformation of the contour in their vicinity reflect alternately from the tangential discontinuity and from the wall, to arrive at the point m. Here it must be remembered that the pressure perturbation is reflected from the wall without change of sign.

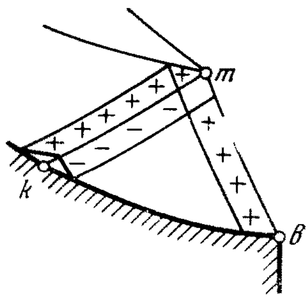

Fig. 2

3. Our next investigation refers to the case when the optimal configuration demands that an expansion fan is generated by the flow at the corner point m (in accordance with what was said before, this can be expected when $K_m > 0$). We shall also limit ourselves to the situation in which the characteristic lk belonging to the second family and arriving at k, intersects ah below the tangential discontinuity, as shown in Fig. 3 (thin lines different from dm represent the characteristics), or at the point f.

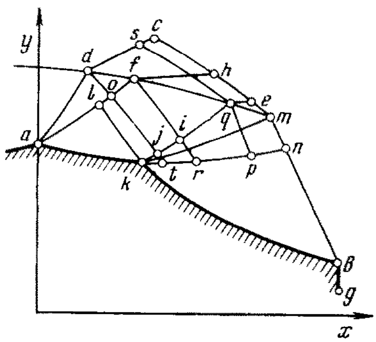

Fig. 3

As we have already remarked, the presence of an internal corner makes it necessary to employ the general method of Lagrange multipliers. First we construct the following auxiliary functional

$$I = \chi + \int_a^b \tau L_3 \, dy + \iint_G (\mu_1 L_1 + \mu_2 L_2) \, d\psi \, dy$$

in which $\tau = \tau(y)$ and $\mu_i = \mu_i(\psi, y)$ are variable Lagrange multipliers and G is the region of flow (in the ψy-plane) bounded by the contour of the body (the straight line $\psi = \psi_a$) and the characteristics ac and cb. By virtue of (1.2) and (1.4) we find, under the allowed variations, that when u and v, as well as the density and pressure which are, in accordance with (1.2) all known functions of u, v and ψ, satisfy the equations and the boundary conditions of the problem, then the first variations of I and of the optimized functional χ, coincide. Moreover, acting in accordance with [3 – 5, 7] we obtain the equations and conditions of the "conjugated" problem which can be used to find the Lagrange multipliers in the subregion G° of G lying (in the xy-plane) to the right of ah and in particular on ab, as well as the necessary conditions for χ to be a minimum, which determine the form of the optimal generatrix.

In the subregions of G° in which the multipliers μ_1 and μ_2 are continuous, the latter must satisfy the following system of equations

$$\frac{\partial\mu_1}{\partial y} + y^\nu\rho u\,\frac{\partial\mu_1}{\partial\psi} + \frac{1}{v}\,\frac{\partial\mu_2}{\partial\psi} + \frac{uv}{y^\nu\rho a^2 v^2}\,\frac{\partial\mu_2}{\partial y} = 0$$

$$y^\nu\rho v\,\frac{\partial\mu_1}{\partial\psi} - \frac{a^2 - v^2}{y^\nu\rho a^2 v^2}\,\frac{\partial\mu_2}{\partial y} - \frac{u}{v^2}\,\frac{\partial\mu_2}{\partial\psi} = 0$$

When $w > a$, the system given above possesses two families of real characteristics which coincide with the characteristics of the equations of motion (1.2) and on which

$$R d\mu_1 \pm d\mu_2 = 0 \qquad (R = y^\nu\rho v^2 / \beta, \quad \beta = \sqrt{M^2 - 1}) \tag{3.1}$$

Here and in the following the upper (lower) sign denotes the characteristics of the first (second) family. The differentials $d\mu_i$ in (3.1) are taken along the characteristic.

On the characteristics which lie in G° and represent the lines of discontinuity of Lagrange multipliers, the jumps in μ_i satisfy the equations

$$R\,[\mu_1] \mp [\mu_2] = 0 \tag{3.2}$$

where $[\mu_i]$ is the difference in the values of μ_i before and after the discontinuity (in the direction of flow). The boundary conditions of the conjugated problem for μ_i are formulated at the wall ab and on the closing characteristic hb and have the form

$$\mu_1 = 1 \text{ on } ab, \qquad R\mu_1 + \mu_2 = 0 \quad \text{on} \quad hb \tag{3.3}$$

Finally, the multiplier τ on ak and kb is determined in such a manner that $\tau = \mu_2$.

On the line of contact discontinuity fm the multipliers μ_i are continuous. On the other hand, during the passage across the tangential discontinuity the coefficients accompanying μ_i in the second condition of (3.3) change at the point m. This implies that the characteristic km is a line of discontinuity of the Lagrange multipliers. The intensity of the discontinuity at the point m characterized e.g. by the jump $[\mu_2]_m$ is determined from (3.3) and (3.2) and is given by the formula

$$[\mu_2]_m = -K_m \mu_{2m+} \tag{3.4}$$

Here and in the following the index minus (plus) accompanying μ_i denotes the values of μ_i before (after) the corresponding line of discontinuity. If the reflection coefficient vanishes at the point m then, as we see from (3.4), the characteristic km ceases to be a line of discontinuity of the multipliers μ_i and this corresponds to the solution without internal corner points discussed earlier.

Use of the second condition of (3.3) enables us to integrate (3.1) which corresponds to the characteristic of the second family, and thus arrive at the formulas for μ_i on hb which are

$$\mu_1 = -CR^{-1/2}, \quad \mu_2 = CR^{1/2} \quad \text{on } hb \tag{3.5}$$

Here C is a constant which has different values on hm and mb, respectively. These values of C are defined from the values of μ_{1b} and μ_{2m-} computed in (3.5) and the values obtained in accordance with (3.3) and (3.4). Thus e.g. on mb, by virtue of (3.3) we have $C = -R_b^{1/2}$.

Taking into account the fact that the characteristics, i.e. the lines of discontinuity of the Lagrange multipliers, satisfy on each of their sides the corresponding differential relation given in (3.1), we arrive in an analogous manner at the formulas

$$[\mu_1] = [\mu_1]_n (R_n / R)^{1/2}, \ [\mu_2] = [\mu_2]_n (R / R_n)^{1/2} \tag{3.6}$$

These formulas define $[\mu_i]$ at any point on the line of discontinuity in terms of R, i.e. in terms of y and the flow parameters at this point, as well as in terms of the intensity of the jump in μ_i and the value of R at any point n on the same characteristic.

For an arbitrary (not necessary optimal) contour akb about which a flow takes place without formation of shock waves at G^c, the equations and boundary conditions (3.1)–(3.6) make it possible to solve the conjugate problem in G° and find, in particular, the values of the multipliers μ_i at the contour akb. At the same time it can be shown that in the case depicted in Fig. 3 the characteristic lk represents another line of discontinuity in μ_i. Further, if the contour considered is optimal, then the multiplier μ_2 retains a constant value on each of its smooth segments

$$\mu_2 = C_1 \quad \text{on } ak, \qquad \mu_2 = C_2 \quad \text{on } kb \tag{3.7}$$

and the following conditions hold at the corner point k

$$E_1 \equiv \int_{k_-}^{k_+} \{R(\mu_1 - 1) + \mu_2 - \mu_{2k+}\} \, d\frac{u}{v} = 0 \tag{3.8}$$
$$E_2 \equiv \mu_{2k+} - \mu_{2k-} = 0$$

Here the integral at the point k is taken across the whole expansion fan and the lower minus (plus) sign denotes the parameters on the wall before (after) the corner. Comparing (3.7) and (3.5) we see that $C_2 = -C^2 = -R_b$. In addition to satisfying (3.7) and (3.8), the optimal contour must satisfy another two conditions of optimality

$$\{y^\nu (p^+ - p) + \mu_2 (u / v)\}_b \geqslant 0, \quad \mu_{2b} \leqslant 0 \tag{3.9}$$

The first of these conditions (with the equality sign) serves to determine y_b and the second represents the condition that the butt end bg is the segment of the edge extremum. From (3.7), with the expression for C_2 taken into account, it follows that the condition always holds provided that $v_b \neq 0$. The inequality in the first condition of (3.9) can only arise when $y_b = y_g$.

Constructing the segments ak and kb of the optimal contour (with the angles by which the direction of the flow is changed at the points a and k, known) is equivalent to determining the "optimal" characteristics lk and nb. The passage to the characteristics indicated, i.e. the transfer of the corresponding conditions of optimality to these characteristics, can be realized due to the fact that in this case, just as in [7, 8], the equations $\mu_1(\psi, y) \equiv 1$ and $\mu_2(\psi, y) \equiv \text{const}$ represent the integral of (3.1) which yields a solution to the Cauchy problems in the triangles alk and knb with the initial conditions (3.3) and (3.7) on ak and kb. As the result, in the course of solving the conjugated problem the Lagrange multipliers need only be determined in the region $lhnkl$ and the boundary conditions for μ_i are set on hn and kn. The conditions on hn follow directly from the previously formulated conditions on hb, and the conditions on kn assume the form

$$\mu_1 = 1, \quad \mu_2 = C_2 \quad \text{on } kn \tag{3.10}$$

The constant C_2 as well as the constant C from (3.5) on the segment mn are better expressed in the terms of the flow parameters at the point n. In accordance with (3.5) and (3.10) we have $C = -R_n^{1/2}$ and $C_2 = -R_n$.

Further, since $\mu_1 \equiv 1$ everywhere in the triangle knb, from (3.5) it follows that the quantity R remains constant on the segment nb. Taking into account the expression for R after the passage from v and β to w, ϑ and α, we arrive at the condition (2.1) obtained with the help of the method of control contour. At the same time the first condition of (3.9) which serves to determine y_b assumes, after substituting μ_{2b}, the form

$$\{\beta(p^+ - p) - \rho uv\}_b \geqslant 0 \tag{3.11}$$

In the case of equality (i.e. when $y_b > y_g$) this condition coincides with (2.3) which was also obtained using the method of control contour. The naturalness of coincidence of the conditions on nb and at the point b with the conditions obtained previously, is obvious. In fact, by virtue of the supersonic character of the flow the segment kb, as well as any end part of the body, must be optimal. Since the tangential discontinuity passes above the point n, it can also be constructed using the method of control contour.

The condition determining the optimal distribution of parameters on the characteristic lk is obtained from (3.2) in the similar manner and is formulated as the following equality

$$E_3 \equiv R(\mu_1 - 1) + \mu_2 - C_1 = 0 \quad \text{on} \quad lk \tag{3.12}$$

which is equivalent to the condition (3.7) on ak. The Lagrange multipliers in (3.12) are taken from the right of the characteristic lk which represents the line of discontinuity of μ_i. The last quantities to be determined in the course of solution of the conjugate problem are μ_{1l} and μ_{2l}. The constant C_1 is then chosen such, that $E_{3l} = 0$.

We can deal in the similar manner with the variational problem of constructing the generatrix of the supersonic part of a Laval nozzle realizing the maximum thrust. Figures corresponding to this problem can be obtained from Figs. 1 – 3 by mirror reflection (without changing the orientation of the coordinate system) in a straight line $y = \text{const} > y_c$. Since in this case the characteristics of the first family become the characteristics of the second family (and vice versa), certain of the relations obtained above are modified. Thus, Eqs. (2.2) and the second equations in (3.3) and (3.5) which hold on the closing characteristic, must be replaced by

$$\frac{dx}{d\psi} - \frac{u\sqrt{w^2 - a^2} - av}{y^\nu \rho a w^2} = 0, \qquad \frac{dy}{d\psi} - \frac{v\sqrt{w^2 - a^2} + au}{y^\nu \rho a w^2} = 0$$

$$u^2 \frac{d(v/u)}{d\psi} + \frac{\sqrt{w^2 - a^2}}{\rho a}\frac{dp}{d\psi} + \frac{\nu v}{y^{1+\nu}\rho} = 0$$

$$R\mu_1 - \mu_2 = 0, \qquad \mu_2 = -CR^{1/2}$$

respectively. The conditions (2.3) and (3.11) at the point b now become

$$(\rho w^2 \operatorname{tg} \alpha \sin 2\vartheta)_b = 2(p_b - p^+), \quad \{\beta (p^+ - p) + \rho uv\}_b \geqslant 0$$

and Eq. (3.12) defining the characteristic lk becomes

$$E_3 \equiv R(\mu_1 - 1) - \mu_2 + C_1 = 0 \quad \text{on } lk$$

The expressions for the constant C_2 in (3.7) and (3.10) are also changed to $C_2 = C^2 = R_b$ and $C_2 = R_n$, respectively. Moreover, in the inner problem it is expedient to assume that $\psi_a = 1$, and choose the normalizing constant c for the stream function in (1.1) so, that the value of ψ on the axis of symmetry becomes zero when $\psi_a = 1$. After this it only remains to replace the expressions of the type "minimum drag" with the words "maximum thrust" to make all previous statements referring to the outer problem, valid for a nozzle.

4. Let us consider a special case of a plane flow ($\nu = 0$), in which $H(\psi)$ and $S(\psi)$ are piecewise constant functions which undergo a discontinuity only when $\psi = \psi_d$ and where the direction of the velocity vector on the segment dc of the initial characteristic does not vary ($\vartheta \equiv \vartheta_d$). The remaining gas parameters are constant on dc and the segment itself is a straight line. For this reason we have, in the quadrangle $dchf$ of the initial fan, a Prandtl-Meyer flow with rectilinear characteristics of the first family.

Let us now assume that ϑ on the optimal characteristic lk determining the form of the initial segment of the generatrix, is also constant (*). Then the characteristics indicated will be rectilinear and so will be, in the quadrangle $lfik$, all characteristics of the same family as lk. Since in the present case $p \equiv p_f$ and $\vartheta \equiv \vartheta_f$ on fi and fh, the segment fq of the line of tangential discontinuity is found, in accordance with the equation of flow, to be rectilinear. The same is true of the segments of all the characteristics of the second family lying in $fheqif$ on one side of the tangential discontinuity (on fq the characteristics of either family have a corner).

Considering further the conjugate problem for the Lagrange multipliers we can show that the latter, under the assumption made above, are constant on each characteristic of the second family $lfheqkl$ and consequently, also on lk. From this it is clear that in the problem in question the constant distribution of ϑ and other parameters on the characteristic lk satisfies (by virtue of the choice of C_1) the condition (3.12), i.e. it is optimal.

This circumstance appreciably simplifies the problem of constructing the optimal configuration by narrowing the region in which the conjugate problem must be solved, to the second expansion fan $kqenk$. Moreover in this case the solution of the inverse problem in which instead of X and p^+ we specify the intensity of the initial fan and the position of the point l on af, the iterative procedure for numerical solution of the problem representing the most time-consuming part of the corresponding algorithm, here involves the determination of two quantities only, the corner angle on the wall at

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*) We note that in the present case this, by virtue of (2.1) and (2.2), certainly takes place on the segment nb of the closing characteristic and on all characteristics of the second family, in the triangle knb. In this case of a plane irrotational flow, all characteristics of the second family are rectilinear in the domain of definition $ahba$ of the contour sought.

the point k and the ordinate (or abscissa) of the point e on ke. These are obtained from the requirement that the condition (3.8) holds during the process of repeated solution of the direct (to compute the flow parameters) and the conjugate (to obtain μ_i) problems in $kqenk$. In the general case, however, the numerical algorithm must include an additional procedure for computing the optimal distribution of $\vartheta = \vartheta(\psi)$ on lk.

If sq is a characteristic of the second family, then the solution obtained above remains valid even when the assumptions formulated in Sect. 4 concerning the character of the flow on dc hold only on the segment ds. The flow is arbitrary on sc (in the case of rotational flow we additionally require that ψ_h does not exceed ψ_s on sc). We also note that in the solution considered the Prandtl-Meyer flow with linear characteristics of the first family is also realized in the rectangle $iqpr$ and, within this rectangle the multipliers μ_i remain constant along the rectilinear characteristics. If we include in the previous assumptions an additional one, that $\vartheta \equiv$ const also on ad, then the triangles ado and kjt are also simple wave regions. In the latter the Lagrange multipliers are constant on the characteristics of the first family.

5. Before passing to the examples illustrating the computations carried out for the configurations of the type shown in Sect. 4, we shall give the results of an analysis of the case of a weak tangential discontinuity. We note that this analysis is valid for any incident flow for which a continuous solution described by (2.1) can be constructed in the absence of a tangential discontinuity.

Thus let the parameters of a continuous ("initial") free supersonic stream and other conditions of the problem be known, together with the quantity ψ_d and the jumps in the values of H and S which occur when $\psi = \psi_d$, the latter being of the order of ε and defining a "perturbed" flow with a tangential discontinuity. It is convenient to assume that the parameters of the initial and the perturbed flow coincide on ad and the appearance of the tangential discontinuity modifies the flow on dc If, as before, km is the characteristic of the first family which arrives at the point m of the closing characteristic when $\psi_m = \psi_d$, then the parturbation of the free stream leads to the appearance of a corner at k and causes deformation of the segments ak and kb of the initial optimal contour. The purpose of the present analysis is to obtain formulas for computing the corner angle at k, i.e. the difference $(\zeta_{k+} - \zeta_{k-})$ as well as the increments $\Delta\zeta_1$ and $\Delta\zeta_2$ corresponding to the parts of the corner before and after the characteristic km respectively. The formulas given above are accurate up to and including ε and are valid for the extreme right inner corner point when any number of such points exist (i.e. in the cases depicted on both, Fig. 3 and Fig. 1).

It is natural to expect that the introduction of a weak tangential discontinuity does not significantly distort the initial optimal contour constructed with $\varepsilon = 0$. We therefore assume that $\Delta\zeta_1$, $\Delta\zeta_2$ and $\zeta_{k+} - \zeta_{k-} = \Delta\zeta_1 + \Delta\zeta_2$ are all of order of ε. The subsequent investigation is based on the equations and conditions obtained above, and includes the following stages. First we construct, in the usual manner, the optimal contour and the complete flow in ahb corresponding to $\varepsilon = 0$, i.e. to a continuous flow at ac. This, with ψ_d given, enables us to find the unperturbed characteristic km and the quantity

$$A = -d \ln R \,/\, d\zeta$$

where $d \ln R$ is the differential computed on the characteristic mb and corresponding to the increment $d\zeta$ at the point k. Both increments are determined at the known characteristic km for the flow in the expansion fan which appears to the right of km when a corner is introduced at the point k.

We know [1, 2, 9] that for any flow, a derivative of the type A can be expressed in terms of the distribution of the gas parameters on km. In the case of a plane irrotational flow when the initial flow is a simple wave, the formula for A becomes appreciably simplified and can be obtained from the expression for R and the condition of compatibility appearing in (2.2). In the general case A can always be defined numerically, using the method of characteristics to compute that characteristic of the fan which is adjacent to km, and utilizing the ratio of the increments of $\ln R$ and ζ at the corresponding points of the initial and the computed characteristic.

Making further use of the quantity A together with the relations (3.5) on hb and (3.4) at the point m, we arrive at the formulas which yield, with the accuracy up to ε, the values of μ_1 and μ_2 on the segments mn and me of the closing characteristic (unlike before, here and in the following we consider the characteristics of the perturbed flow). As the width of the fan $kqemnk$ and, consequently, of its parts lying on the left and right of km are of order of ε, we use Eqs. (3.1) defining the characteristics for μ_i as well as the conditions $\mu_1 \equiv 1$ and $\mu_2 \equiv \mu_{2k+}$ on kn and the formulas (3.6) at the discontinuity km, we perform the necessary integration to define μ_i in ζ at the point k, making use of the already obtained distributions of μ_i on mn and me. The relations indicated have, with the accuracy of up to and including ε, the form

$$(\mu_1 - 1)(R_k / R_m)^{1/2} = 0.5A(\zeta - \zeta_{k+}) + \gamma K_m$$
$$(\mu_2 - \mu_{2k+})(R_m R_k)^{1/2} = 0.5A(\zeta - \zeta_{k+}) + \gamma K_m$$

where $\gamma = 0\ (1)$ to the right (left) of km, while the quantities R_m and R_k are computed for an unperturbed optimal profile.

Inserting these distributions into the conditions (3.8) and performing the integration with respect to ζ in the first of the resulting expressions we obtain two expressions for $\Delta\zeta_1$ and $\Delta\zeta_2$ which on solving yield

$$\Delta\zeta_1 = \Delta\zeta_2 = K_m / A \tag{5.1}$$

From the above formula it follows that for small values of the coefficient of reflection the corner angle in the contour is in fact proportional to $K_m \sim \varepsilon$, and the characteristic km divides the contour in half. When A is negative (this occurs in all cases considered here and apparently, always), the above argument together with the inference made previously imply that for the positive coefficients of reflection at the point m, the flow past the corner at k forming an expansion fan just as we assumed in our previous analysis. It is obvious that the case $K_m < 0$ cannot be analized in the same manner.

6. The linear analysis given in Sect. 5 can only be used when K_m are small. When the coefficients of reflection are moderate or large (by definition $|K| \leqslant 1$), the required information can only be obtained by numerical method. The most interesting problems here are the following: (1) what is the magnitude of the corner angle at k and (2) how much smaller is the wave drag of the optimal contour, than the wave drag of a contour constructed by any other (simpler) method. With regard to the first problem we note that the corner angles at the internal points of the head-end contours of bodies with minimal wave drag, the flow past accompanied by an attached shock wave, are found to be very small in all cases investigated [10, 11].

To obtain the answer to the above questions we have derived the necessary computer algorithms and used them to construct the optimal contours discussed in Sect. 4. The whole free stream was assumed parallel to the x-axis, i.e. $\vartheta \equiv 0$ on ac, and the remaining gas parameters of this flow were assumed constant on either side of the tangential discontinuity. A flow of perfect gas was considered, with the adiabatic index $\varkappa = 1.4$ for the whole stream. The inverse problem was solved in which, as we said before, instead of specifying the maximum allowed length X and p^+, we specified the intensity of the corner at the initial point coincident with coordinate origin ($y_a = x_a = 0$) and the position of the point l on the characteristic ah. For this reason, instead of the length of the body, we used the ordinate of the line of tangential discontinuity in the free stream ($y_d = 1$) as the characteristic dimension (l_*). The critical velocity and density of the unperturbed flow in a layer adjacent to the wall were used as the characteristic velocity and density w_* and ρ_*. Therefore the ratio $p / \rho^{\varkappa}$ used in place of entropy, and the stagnation enthalpy, are

$$S_+ \equiv \left(\frac{p_\infty}{\rho_\infty{}^{\varkappa}}\right)_+ = \frac{1}{\varkappa}, \qquad H_+ \equiv \left(h_\infty + \frac{w_\infty{}^2}{2}\right)_+ = \frac{\varkappa + 1}{2(\varkappa - 1)}$$

Here the subscripts ∞ and a plus (minus) are assigned to the parameters of the free stream past the wall (above the tangential discontinuity).

The computations were performed using the method of characteristics and included solving the direct problem to obtain the flow parameters, and the conjugate problem to obtain the Lagrange multipliers. Solution of the direct problem involves a successive determination of the initial fan $adchfa$, the region $klfheqk$ and of the second fan $kqemnk$. The flow in $klfheqk$ is computed along the characteristic ah obtained in the process of constructing the initial fan, and along the rectilinear characteristic lk the parameters on which coincide, in the present case, with their values at the point l. The conjugate problem presupposes that the equations of flow and the equations for μ_i are integrated only in the second fan (the computation of the first two regions is only tentative, since the position of the point c and, consequently, the whole characteristic ce, are not known in advance and are only found in the course of solution). In the conjugate problem the computations are performed in the reverse direction (from the characteristic kn). First a new characteristic of the first family is constructed, beginning at the point k. Then the parameters on this characteristic and on the previous characteristic together with the boundary conditions for μ_i on ne are used to find the Lagrange multipliers. Unlike the flow parameters, here the multipliers μ_i at the point k are the last to be determined on each characteristic of the fan. In computing the multipliers we use the equations of characteristics (3.1) with the conditions on kn and ne, and the relations at the discontinuity km.

In the process of finding μ_i at a corner point (on various characteristics of the fan) we compute the integral in (3.8). This enables us, after finding the solution to the conjugate problem, to determine the left hand sides of both conditions in (3.8), the second one of them being rewritten for convenience in the form

$$E_2 \equiv R_k(\mu_{1k} - 1) + (\mu_{2k} - \mu_{2k+})$$

where the subscript k with the plus sign omitted denotes the quantities at the point k on the characteristic ke.

In the present case (and always, when the characteristic lk is fixed) the left hand parts of the equations examined, i.e. E_1 and E_2, are functions of $\Delta\zeta_1$ and $\Delta\zeta_2$ only, and they are different from zero when the choice of the indicated increments is not optimal (the fact that E_1 and E_2 are not explicitly related to $\Delta\zeta_1$ and $\Delta\zeta_2$ but are obtained in the course of computation, is unessential). The optimal values of $\Delta\zeta_i$ are the roots of the equations

$$E_1(\Delta\zeta_1, \Delta\zeta_2) = 0, \qquad E_2(\Delta\zeta_1, \Delta\zeta_2) = 0$$

which, as in [10, 11], were obtained in the course of the iterative process using the Newton's method. Each iteration here includes the determination of partial derivatives of E_i with respect to $\Delta\zeta_1$ and $\Delta\zeta_2$ and requires a three-stage solution of the direct and conjugate problem for the second fan. The initial approximation for $\Delta\zeta_i$ was either obtained using the formula (5.1) of the linear theory, or was taken from a previously computed version. In all cases considered here the number of iterations guaranteeing the correctness of all significant figures in the results given below, did not exceed four.

After completing the iterations over $\Delta\zeta_i$ we project, from the point n with $\psi=\psi_n$, an optimal characteristic nb which is in this case rectilinear (the parameters on nb are constant and equal to those at n). The coordinates of the point b as well as $X = x_b$ are found, along $\psi_b = \psi_a$, on this characteristic. The counter pressure p^+ is found from the Busemann condition (2.3). If the value $p^+ \geqslant 0$ is obtained, then for the given free stream and for X and p^+ which have been determined and can therefore be now assumed given, the configuration constructed is optimal for any $y_g \leqslant y_b$. The segments ak and kb of the generatrix ab are subsequently obtained as the streamlines, with $\psi = \psi_a$, from the solution of two Goursat problems in which the characteristics al, lk, kn and nb, respectively, are known. The wave drag χ_{ab} of the segments ak and kb can be found either by direct integration along the contour ab, or (prior to constructing the segments ak and kb) using the momentum flows across the segments al, lk, kn and nb of the known characteristics. The difference in the values obtained by each method is usually employed in assessing the accuracy of the computations.

After constructing each optimal configuration we set up, for comparison, a nonoptimal contour corresponding to the same free stream with given (identical to those already obtained) values of X and p^+. The nonoptimal contours are chosen such, that the closing characteristics $h^\circ b^\circ$ of their domain of definition satisfy the condition (2.1) with a separate value of the constant λ on each segment, $h^\circ m^\circ$ and $m^\circ b^\circ$. On $h^\circ m^\circ$ the constant λ is equal to the left-hand part of (2.1) at the point h° and on $m^\circ b^\circ$, to the same expression at $m_+{}^\circ$. The parameters at m_+° are found from the parameters at $m_-{}^\circ$ by virtue of the conditions of continuity at p and ζ on the tangential discontinuity. The intensity of the initial fan and the position of the point h° on its closing characteristic are chosen so, that the contour constructed is of the prescribed length X and the Busemann condition (2.3) holds at its end point b° when a known pressure p^+ acts on its butt end. It can be shown that in this case the method of constructing a comparison contour ensures the optimality not of the contour ab° as a whole, but of each of its segments ak° and $k^\circ b^\circ$ separately.

In the linear analysis of the conjugate problem carried out in Sect. 5 the linearization was performed with respect to the solution corresponding to a profile which becomes optimal when the free stream contains no tangential discontinuity. A different approach can also be adopted, in which the role of the initial flow is played by a flow past a smooth, nonoptimal contour, corresponding to the previously constructed characteristic

$h^\circ b^\circ$. In this case the initial ("unperturbed") flow already contains the tangential discontinuity. In spite of this discrepancy, the final result obtained in this case coincides with the formula (5.1) in which the quantity A can now be computed on either side of the characteristic $k^\circ m^\circ$ of the initial flow. It was precisely in this manner that the corner angles given below for comparison were computed from (5.1), with A taken as equal to half of the sum of the corresponding quantities in m_-° and m_+°.

Several optimal contours were computed and their characteristics are given in Table 1.

Table 1

$X =$	3.30	3.45	5.10	7.61	11.3
$p^+ \cdot 10 =$	3.87	3.71	2.01	0.81	0.12
$x_k =$	1.41	1.41	1.91	2.60	3.08
$-y_k \cdot 10 =$	0.70	0.68	1.85	3.43	5.04
$-y_b =$	0.17	0.19	0.58	1.39	3.09
$-y_g =$	0.19	0.22	0.67	1.66	4.00
$-\zeta_a \cdot 10^2 =$	5.00	5.00	10.0	15.0	19.0
$-\zeta_{k-} \cdot 10^2 =$	5.00	3.40	6.08	7.40	9.10
$-\zeta_{k+} \cdot 10^2 =$	5.12	7.39	1.51	2.35	3.59
$-\zeta_b \cdot 10^2 =$	5.13	4.59	9.55	1.57	2.41
$-\Delta\zeta_1 \cdot 10 =$	0.06	0.22	0.50	0.89	1.39
$-\Delta\zeta_2 \cdot 10 =$	0.06	0.18	0.40	0.72	1.28
$-\Delta\zeta_i \cdot 10 =$	0.07	0.10	0.20	0.31	0.41
$-\zeta_{a^\circ} \cdot 10^2 =$	6.16	9.05	19.4	32.7	50.8
$-\zeta_{b^\circ} \cdot 10^2 =$	5.02	3.47	7.08	11.6	20.5
$-\chi \cdot 10^2 =$	8.56	10.0	22.2	35.7	46.9
$\Delta\chi \cdot 10^2 =$	0.95	1.88	3.18	4.57	3.45

Each column gives, in addition to the quantities pertaining to the optimal contour, some of the geometrical characteristics of the smooth comparison contour, the quantity $\Delta\zeta_i$ which is found from the formula (5.1) of the linear theory and the gain in χ due to the optimal profiling. The latter is given by the relative increment $\Delta\chi \equiv (\chi - \chi^\circ) / \chi^\circ$, where χ° is the wave drag of the nonoptimal configuration (by definition, χ and χ° are negative). The ordinates y_g and y_{g° of the end points of the optimal and the comparison

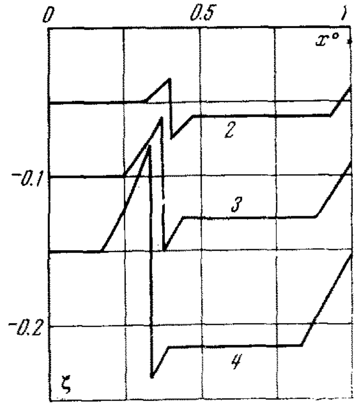

Fig. 4

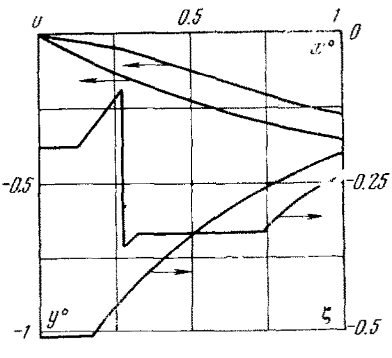

Fig. 5

contour, respectively, were assumed both equal to y_{b°. The parameters of the free stream in the case of the first contour were: $S_- = 7.143$, $H_- = 30$, $M_- = 2.73$, $M_+ = 1.2$ and $p_+ = 0.556$, which gives $K_\infty = 0.15$. For the remaining four contours in the same unperturbed stream the values obtained were: 0.143, 12, 8.57, 1.1, 0.633 and 0.53, respectively.

Figure 4 shows the distributions of ζ along three of the contours constructed. Here $x^\circ = x / X$ and the numbers accompanying the curves refer to the contours. An analogous curve for the fifth optimal contour, and its form, i.e. the relationship connecting $y^\circ = y / X$ and x°, are shown in Fig. 5, which also depicts the distribution of ζ and the smooth contour itself, corresponding to the same X, p^+ and the free stream as the contour №5. In contrast to the remaining smooth contours computed for comparison purposes, the flow past the smooth contour shown in Fig. 5 is such, that the closing characteristic of its domain of definition does not intersect the tangential discontinuity. Since all conditions of optimality hold for such smooth contour, the variational problem considered here has two solutions (see also Sect. 7). Comparing the values of χ achieved by the contours constructed shows that in the example given above the contour with an internal corner point is superior to the smooth optimal contour.

The results given above show, that in the present problem (in contrast to the situation observed for the bow parts of the bodies with minimum drag [10, 11]) the corner at k is of the same order, or even exceeds, that at the initial point, i.e. ζ_a. We note also that despite the fact that $\Delta\zeta_1$ and $\Delta\zeta_2$ are different from $\Delta\ \zeta_i$, the ratio $\Delta\zeta_1 / \Delta\zeta_2$ is, in all cases, nearly equal to unity, as predicted by the linear theory.

7. In conclusion we shall dwell on two problems which are interesting in their own right. First we shall follow the evolution of the optimal configuration when X is varied over the range of all possible values. To do this we fix all conditions of the problem except X, i.e. except the free stream and p^+, and vary X step by step, beginning from zero and gradually increasing its value. Since for sufficiently small X, the closing characteristic hb lies completely between the tangential discontinuity and the wall, therefore the usual configuration containing no internal corner points, is an optimal one. As X is increased, the point h recedes from the point a and at a certain instant (when $X = X_1$ where X_1 is a function of p^+ and of the parameters of the free stream) it becomes possible to construct an optimal configuration of the type investigated above. Since at the point k both conditions of (3.8) must hold and, as in the problem on the compound nozzle [7], the appearance of the second corner takes place from "splitting" the initial fan into two fans (coinciding at the instant of splitting) of finite intensity, situated close to each other. We must stress that this occurs before the point h, corresponding to a solution without a corner, reaches the tangential discontinuity. Therefore, over a cetrain range of maximum allowed lengths, two solutions exist (with and without a corner). It is not known in advance which of these solutions gives a smaller χ, therefore the optimal contour must be chosen in each case (when two solutions exist) by comparing the values of χ just as it was done in Sect. 6.

Increasing the value of X further, we pass to the cases of two, three, etc. internal corner points, each passage accompanied for the reasons given above by splitting of the fan at the point a and the appearance of a two-solution region.

Let us consider the case of large X from another point of view. Let the length become such, that the situation depicted in Fig. 1 obtains for the smooth contour. Here a polygonal curve composed of segments of the characteristics and terminating at the point b is reflected, within the domain of definition, from the tangential discontinuity, more than once. We shall restrict ourselves to the case when $K > 0$ at the points of reflection. We also know that the coefficient of reflection from the solid wall, introduced here in the analogous manner, is equal to unity. Therefore an analysis similar to that conducted in the final part of Sect. 2 shows that, when the smooth contour is deformed in the manner depicted in Fig. 2 at any of the points k, $k_1, \ldots$, then the value of χ is reduced. This gives some grounds for the assumption that when $K > 0$, the flow past all internal corners will lead to formation of an expansion fan at each corner.

The second problem which is particularly interesting, concerns the deformation of the optimal contour, when the tangential discontinuity becomes continuously diffused. At the first glance, a corner appears at the point k because the coefficient of reflection at the point m is not zero. In the case of a smooth contour this leads, by virtue of the conditions of the conjugate problem, to formation of a discontinuity in μ_i on the characteristic mk and consequently to the necessary appearance of a corner at the point k. It is the reflection of perturbations from the shock wave [10, 11] and from the axis or plane of symmetry [12, 13] that is the basis of the mechanism of appearance of the internal corners in other problems (here of course the corners which are not connected with the process of specifying the coordinates of an internal point which was the case in [14], or with the different role played by the corresponding segments of the generatrix already in the initial functional, as it was in the case concerned with the point at which the initial segment meets the end segment of a compound nozzle [7]). If the tangential discontinuity becomes diffused, i.e. if it is replaced by a narrow zone of continuous variation of parameters and the reflecting surface dissapears, then the jump in the values of the Lagrange multipliers which occurred earlier on km, also becomes diffused between the characteristics k_-m_- and k_+m_+. At the same time the pattern of reflection of perturbations becomes more complex, and this makes it more difficult to analyse the situation with help of a method which made it possible for the case of tangential discontinuity in Sect. 2 to establish without difficulty the necessity of the appearance of a corner on the contour. However, we shall show below that a weak blurring of the tangential discontinuity does not lead to disappearance (or to a weak "rounding") of the internal corner point of the optimal contour.

Let us assume the opposite, i.e. suppose that replacing the line of discontinuity dm by a narrow zone of continuous variation of parameters bounded by the streamlines d_-m_- and d_+m_+ causes a weak (or strong) rounding of the corner at the point k of the generatrix which is optimal when $\Delta = 0$, where Δ is the width of the region of sharp variation of parameters. For any smooth generatrix a solution of the problem can be obtained using the method of control contour, i.e. reduces to the results of Sect. 2. Thus the optimal distribution of the parameters everywhere on hb, and in particular on m_-m_+, must satisfy the condition (2.1) with the constant λ computed using the value of the left-hand side of (2.1) at the point h. However, as soon as the distribution of the parameters on m_-m_+ has been found in accordance with (2.1) and the equations of the characteristic of the second family, i.e. (2.2), another question arises concerning the possibility of realizing the optimal distribution obtained by profiling the generatrix ab to the right of the characteristic k_-m_-, with the initial segment of the generatrix (ak_-) fixed. We stress that the corresponding optimal distribution can be constructed for any

Δ and this includes $\Delta = 0$. If $K_m > 0$, then in the latter case the centered rarefaction wave from the characteristics of the first family with the center at the point m corresponds to the discontinuous distribution indicated. This possibility has not been discussed before, because such a flow cannot be realized in any manner by profiling a wall at a finite distance from the point m.

An analogous situation arises when the discontinuity is weakly blurred. In fact, in this case the rate of variation of such parameters as p and ϑ in the distribution obtained from (2.1) (i.e. the magnitude of the corresponding derivatives along m_-m_+) is determined by Δ, becoming infinite when $\Delta = 0$, and the maximum rate of "dispersal" of p and ϑ in an actual flow occurring, as we know [1, 2, 9], at the corner k of the contour ab, is always finite. Consequently, for sufficiently small Δ, the continuous distribution of (2.1) which follows from the assumption that the corner point has disappeared, cannot be realized and therefore the optimal contour must have a corner at k. Thus in the present case the necessity of introducing a corner is dictated by the impossibility of constructing a smooth optimal generatrix.

It is reasonable to expect that weak blurring of the tangential discontinuity will cause a small deformation of the optimal configuration constructed for $\Delta = 0$ with compulsory retention of a corner at the point k. However, for the above assertion to be true it is necessary that the blurring of the discontinuity dm must not appreciably affect the flow parameters and the Lagrange multipliers anywhere outside the narrow zones of width of the order of Δ. When the latter zones are intersected, the relations which are valid for $\Delta = 0$, must hold at the corresponding discontinuities. This requirement is definitely fulfilled for the gas parameters. For the Lagrange multipliers the problem is more complicated. On one hand, on passage through the narrow zones of sharp variation of μ_i bounded by the characteristics of the same kind, the relations (3.2) for $[\mu_i]$ still hold in the regions of smooth variation of the flow parameters (now with the accuracy of up to Δ). The latter can be proved by integrating, across the region indicated, the equation from (3.1) corresponding to the characteristic of the opposite family and determining $[\mu_i]$ as the difference in the values of μ_i on both sides of the zone in question.

On the other hand, as soon as the distribution of parameters on the closing characteristic ceases to be discontinuous, formulas (3.5) for μ_i become valid everywhere on hb and (which is particularly important) the constant C assumes a single value (instead of different values on hm_- and m_+b which occurs when $\Delta = 0$). From this it is clear that the distributions of μ_1 and μ_2 on the segment hm_- of the closing characteristic differ, for any $\Delta \neq 0$, by amounts of the order of unity, from the distributions obtained for $\Delta = 0$. It would seem that this, in turn, must lead to vastly different values of μ_i over the whole region $afhm_-ka$, i.e. to violation of the requirement formulated above which is necessary for the continuous deformation of the optimal contour at small Δ. In fact, none of this takes place. A more detailed analysis shows, at small Δ, another narrow region of sharp variation of μ_i. This region is adjacent to the segment hm_- of the closing characteristic and is bounded from the left by the characteristic $h^\circ m$ of the second family.

In Fig. 6, where f_-m_- and f_+m_+ are the streamlines bounding the region of blurred tangential discontinuity, the zones of sharp variation of μ_i are represented by the dense network of characteristics of the corresponding family. In the triangle mm_-h_- the variations in μ_i along the characteristics of the second family diminish on approaching

h_-m and become of the order of Δ on h_-m. The existence of this region ensures that on $h^\circ m$ in contrast to hm_-, the values of the multipliers μ_i differ from those obtained for $\Delta = 0$ on hm_- by amounts of the order of Δ. This secures the continuous deformation of the optimal configuration when the tangential discontinuity is weakly blurred.

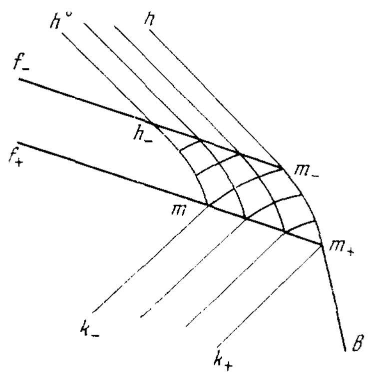

Fig. 6

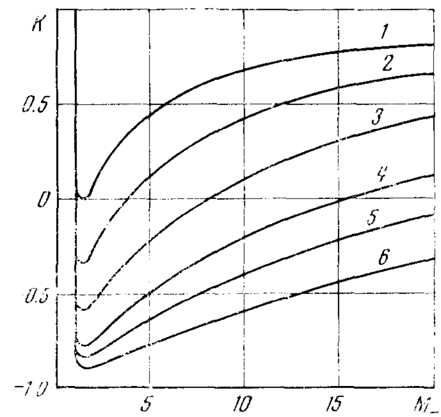

Fig. 7

All the above mentioned points, and in particular the presence in the domain of definition of the contour ab of zones of sharp variation of μ_i, must be taken into account when performing numerical solution of the problem under consideration. We note, by the way, that since the discontinuity in μ_i on the characteristic lk is caused by the presence of a corner in the contour at the point k, it does not become blurred when $\Delta \neq 0$.

The configuration discussed above is realized only when $K_m > 0$, therefore it is important to know the region of the flow parameters, for which the above condition in fact holds. It can be shown that for a perfect gas (when $\varkappa_- = \varkappa_+$)

$$\omega = M_+^2 \sqrt{M_-^2 - 1} / (M_-^2 \sqrt{M_+^2 - 1})$$

where M_- and M_+ are taken at the point in question on the tangential discontinuity. The results of computing $K = K(M_-, M_+)$ are shown in Fig. 7. The curves $1-6$ correspond to the following values of

$$M_+ : \sqrt{2}\,(\sqrt{2}),\ 4\ (1.0328),\ 8\ (1.0079),\ 16\ (1.002),\ 24\ (1.0009)\ \text{and}\ 40\ (1.0003),$$

respectively. The appearance of two values is explained by the fact that each curve corresponding to $M_+ = A$ also corresponds to $M_+ = A/\sqrt{A^2 - 1}$. In accordance with the expression for K and ω the coefficient of reflection vanishes when $M_- = M_+$ and when $M_- = M_+ / \sqrt{M_+^2 - 1}$. For a fixed $M_+ \neq \sqrt{2}$ this gives two points of intersection of the curve $K = K(M_-)$ with the abscissa.

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